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Shear Flow of a Granular Material

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The shear flow of a granular material between parallel plates is treated by means of the Boltzmann equation with pseudo-Maxwellian grains. The moments for reverse reflection boundary conditions are found explicitly. The shearing stress is found to depend quadratically on the shear rate.

KEY WORDS: Granular material; Boltzmann equation; shear flow.

1. INTRODUCTION

In the last few years one has witnessed a notable development of the study of the mechanics of granular materials, because of their growing importance in the applications (sands, powders, rock and snow avalanches, landslides, grains, fluidized beds). The problems related to the study of fast flows of grain materials, which arise, more and more frequently, in industrial processes and are of growing importance in the study of natural phenomena, have been the object of much attention and have been treated with various methods that differ in rigor and complexity. In the majority of these studies one adopts the assumption of one-dimensional flow and neglects the interaction between grains and air. The various methods applied to these simplified problems have also been used to model other important cases of collisional granular motion, including fluidized beds. Some recent work in the field[10–12] deals with the quasi-static flow regime, usually characterized by relatively large densities and prolonged contacts between the grains, as well as more than two-body interactions. Here we are interested in the rapid granular flow.[3, 13] The methods used in this regime include: (i) Development of physical and experimental models (3); (ii) computer simulations; (iii) kinetic theory.

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Actually many recent studies are based on the assumption that, in certain conditions of motion, collisions between particles supply the main mechanism of momentum and energy exchange. This assumption spontaneously suggests an analogy with the kinetic theory of gases. In this theory the particles are of course molecules and there are thus essential differences between the two situations, that must be duly taken into account. In particular, the intermolecular collisions are frequently elastic, whereas this is not a reasonable assumption when dealing with particles of a granular material.

Equations derived for rapid granular flows may be of some use in the quasi-static flow regime, since the dissipative nature of the particle interactions is a feature common to all regimes of granular flow. A direct analysis based on the Boltzmann equation is due to Goldshtein and Shapiro;[9] a more systematic approach has been provided by Sela and Goldhirsch.[14]

In the present paper, following previous work,[1,4] we consider pseudo-Maxwellian particles approximating dissipative hard spheres, with the aim of studying the shear flow of a granular material between two parallel plates. Our equations describe the system of the aforementioned particles undergoing interparticle inelastic collisions, described by a Boltzmann-like collision term, with the boundary conditions of reverse reflection. In the case of a gas this scheme[5,6] produces solutions growing exponentially in time, first found by Galkin[8] and Truesdell[15] for Maxwell molecules and shown to exist for arbitrary models by the author.[7] As indicated in refs. 5 and 6, this time growth is due to the lack of a dissipative mechanism for the heat produced by the shear stress. This mechanism is present in the case of granular material and a steady solution can be obtained.

The paper is organized as follows. In Section 2 we recall our basic equation, and obtain the form of this equation for homoenergetic flows of a granular material. In Section 3 we derive exact equations for the second order moments and show that an exact steady solution exists. This produces a constitutive equation showing that the shear stress is proportional to the square of the strain rate.

2. THE KINETIC EQUATION AND ITS FORM FOR HOMOENERGETIC FLOWS

Let $f(\mathbf{v}, t)$ be a distribution function (here $\mathbf{v} \in \mathbb{R}^3$ and $t \in \mathbb{R}_+$ denote the velocity and time variables, respectively) of a spatially homogeneous system of inelastic particles. Following ref. 1 we describe the system by the pseudo-Maxwellian kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} = B(\rho, t)\, Q(f, f) \tag{2.1}$$

where the term in the right-hand side corresponds to inelastic collisions between particles. The explicit form of the first term is given by the following formulas which correct the strong form of the pseudo-Maxwellian collision integral given in ref. 1:

$$Q(f, f) = \frac{1}{4\pi} = \int_{\mathbb{R}^3} \int_{S^2} [f(t, \mathbf{v}_*) f(t, \mathbf{w}_*) J - f(t, \mathbf{v}) f(t, \mathbf{w})] \, d\mathbf{n} \, d\mathbf{w} \tag{2.2}$$

where $\mathbf{v}_*$, $\mathbf{w}_*$ are the pre-collisional velocities associated to the collision mechanism

$$\mathbf{v}_* = \frac{1}{2}(\mathbf{v} + \mathbf{w}) - \frac{1-e}{4e}(\mathbf{v} - \mathbf{w}) + \frac{1+e}{4e} |\mathbf{v} - \mathbf{w}| \, \mathbf{n} \tag{2.3}$$

$$\mathbf{w}_* = \frac{1}{2}(\mathbf{v} + \mathbf{w}) + \frac{1-e}{4e}(\mathbf{v} - \mathbf{w}) - \frac{1+e}{4e} |\mathbf{v} - \mathbf{w}| \, \mathbf{n} \tag{2.4}$$

and J is the Jacobian of the transformation:

$$J = \frac{1}{e^2} \frac{|\mathbf{v} - \mathbf{w}|}{|\mathbf{v}_* - \mathbf{w}_*|} \tag{2.5}$$

Here $0 < e \leqslant 1$ is the restitution coefficient ($e = 1$ for elastic collisions).

Then the weak form of the collision integral coincides with the weak form given in ref. 1:

$$\int d\mathbf{v} \, g(\mathbf{v}) \, Q(f, f)$$
$$= \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} f(t, \mathbf{v}) \, f(t, \mathbf{w}[g(\mathbf{v}') + g(\mathbf{w}') - g(\mathbf{v}) - g(\mathbf{w})] \, d\mathbf{v} \, d\mathbf{n} \, d\mathbf{w} \tag{2.6}$$

where $g(\mathbf{v})$ is a test function and $\mathbf{v}'$, $\mathbf{w}'$ are post-collisional velocities given by

$$\mathbf{v}' = \frac{1}{2}(\mathbf{v} + \mathbf{w}) + \frac{1-e}{4}(\mathbf{v} - \mathbf{w}) + \frac{1+e}{4} |\mathbf{v} - \mathbf{w}| \, \mathbf{n} \tag{2.7}$$

$$\mathbf{w}' = \frac{1}{2}(\mathbf{v} + \mathbf{w}) - \frac{1-e}{4}(\mathbf{v} - \mathbf{w}) - \frac{1+e}{4} |\mathbf{v} - \mathbf{w}| \, \mathbf{n} \tag{2.8}$$

We denote

$$\rho = \int_{\mathbb{R}^3} f(\mathbf{v}, t)\, d\mathbf{v}, \qquad \rho\mathbf{u} = \int_{\mathbb{R}^3} \mathbf{v} f(\mathbf{v}, t)\, d\mathbf{v}, \qquad 3\rho\theta = \int_{\mathbb{R}^3} |\mathbf{v}-\mathbf{u}|^2 f(\mathbf{v}, t)\, d\mathbf{v} \tag{2.9}$$

where $\rho \in \mathbb{R}_+$, $\mathbf{u} \in \mathbb{R}^3$, and $\theta \in \mathbb{R}_+$ are the density, bulk velocity and temperature of the granular material. Then

$$B(\rho, t) = B(\rho)\sqrt{\theta} \tag{2.10}$$

where $B(\rho)$ is a given positive function of the density ρ (see refs. 1 and 4 for details).

Thus, all notations in (2.1) are explicitly given in (2.2)–(2.10).

Let us now consider a granular material with average density ρ_0, in motion between two parallel plates located at $x = 0$ and $x = L$, respectively. The upper plate moves with velocity V, whereas the lower one is at rest.

We are going to look for a solution which is self-similar, in the sense that if we cut the slab at $x = L'$ and imagine of putting a wall there, the solution in the slab between $x = 0$ and $x = L'$ remains the same, provided we give the plate at $x = L'$ the velocity $V' = VL'/L$. Then the basic parameter must depend on the ratio $K = V/L$ and should be the product of the previous two parameters. We complete our formulation with the boundary conditions: at the plates we assume that the grains satisfy the bounce-back boundary condition in the reference frame of each plate and hence write:

$$f(t, 0, y.\mathbf{v}) = f(t, 0, y, -\mathbf{v}) \tag{2.11}$$

$$f(t, L, y, \mathbf{v}) = f(t, L, y, 2V\mathbf{j} - \mathbf{v}) \tag{2.12}$$

Following a paper by the author,[(7)] we shall look for solutions such that the variable $\mathbf{x}$ appears in f only through the bulk velocity

$$f = f(\mathbf{c}, t) \tag{2.13}$$

where $\mathbf{c} = \mathbf{v} - \mathbf{v}$ is the random velocity.

Homoenergetic affine solutions for the moments of a Maxwell gas were first found by Galkin and turned out to be homoenergetic dilatations.[(8)] The book by Truesdell and Muncaster[(16)] gives a unified discussion of homoenergetic affine flows for a general medium. The defining properties are the following:

(a) The body force (per unit mass) $\mathbf{X}$ acting on the particles is constant:

$$\mathbf{X} = \text{const.} \tag{2.14}$$

(b) The central moments

$$p_{i_1 i_2 \cdots i_n} = \int_{R^3} c_{i_1} c_{i_2} \cdots c_{i_n} f \, d\mathbf{v}; \qquad (i_k = 1, 2, 3) \tag{2.15}$$

are space-homogeneous

(c) The bulk velocity $\mathbf{v}$ is an affine function of position $\mathbf{x}$:

$$\mathbf{v} = \mathsf{K}(t)\,\mathbf{x} + \mathbf{v}_0(t) \tag{2.16}$$

An analysis of the momentum balance equation based on (a), (b), and (c) immediately leads to the following restrictions on K and $\mathbf{v}_0$:

$$\begin{gathered} \dot{\mathsf{K}} + \mathsf{K}^2 = 0 \\ \dot{\mathsf{K}}\mathbf{v}_0 + \mathsf{K}\mathbf{v}_0 = \mathbf{X} \end{gathered} \tag{2.17}$$

The general solution of this system is:

$$\begin{aligned} \mathsf{K}(t) &= [\mathsf{I} + t\mathsf{K}(0)]^{-1}\,\mathsf{K}(0) \\ \mathbf{u}_0(t) &= [\mathsf{I} + t\mathsf{K}(0)]^{-1}\,[\mathbf{v}_0(0) + t\mathbf{X} + \tfrac{1}{2}t\mathsf{K}(0)\,\mathbf{X}] \end{aligned} \tag{2.18}$$

where I is the 3×3 identity matrix. This solution exists globally for $t > 0$ if the eigenvalues of $\mathsf{K}(0)$ are nonnegative; otherwise the solution ceases to exist for $t = t_0$, where $-t_0^{-1}$ is the largest, in absolute value, among the negative eigenvalues of $\mathsf{K}(0)$.

In particular, if

$$[\mathsf{K}(0)]^2 = 0 \tag{2.19}$$

then $[\mathsf{I} + t\mathsf{K}(0)]^{-1} = \mathsf{I} - t\mathsf{K}(0)$ and therefore $\mathsf{K}(t)$ is independent of time. $\mathbf{v}$ is then steady if and only if

$$\mathsf{K}(0)\,\mathbf{X} = 0 \tag{2.20}$$

and $\mathbf{v}_0(0)$ is chosen in such a way that

$$\mathsf{K}(0)\,\mathbf{v}_0(0) = \mathbf{X} \tag{2.21}$$

In particular, this is always possible if $\mathbf{X} = 0$.

Equation (2.19) is satisfied if and only if a coordinate system exists for which the matrix representation of $\mathsf{K}(0)$ is given by

$$((K_{ij}))=\begin{pmatrix} 0 & 0 & 0 \\ K & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.22}$$

For a proof see ref. 7.

When (2.19) applies one talks, for obvious reasons, of a homoenergetic shear flow. The density is not only space-homogeneous but also constant in time for this flow.

Although here we are interested in finding solutions and not in just proving their existence, we mention that solutions of the Boltzmann equation for homoenergetic affine flows exist, even for models more general than Maxwell's. In particular one can show[7] that $f(\mathbf{c}, t)$ must satisfy the following equation

$$\frac{\partial f}{\partial t}-\frac{\partial f}{\partial \mathbf{c}}\cdot \mathsf{K}\mathbf{c}=Q(f, f) \tag{2.23}$$

and prove that this equation admits the following

Existence Theorem.[7] There exists a solution f of the Boltzmann equation, where the cross section in the collision term $Q(f, f)$ does not grow more than linearly in $\mathbf{v}$, $\mathbf{v}_*$ and the initial mass density, energy density, and H-functional $(=\int f \log f \, d\mathbf{v})$ are finite at time 0. These quantities remain bounded for $0 \leqslant t \leqslant \bar{t}$. The time $\bar{t}$ is arbitrary provided $\mathsf{K}(0)$ has no negative eigenvalues. If $\mathsf{K}(0)$ possesses negative eigenvalues and t_0^{-1} is their largest absolute value, then $\bar{t}$ must be smaller than t_0.

Although this was proved for energy conserving collisions, the result should hold for the case of granular materials, although modifications in the proof are certainly required.

3. THE MOMENT EQUATIONS AND THE EXPLICIT SOLUTION

The moments of the distribution function are particularly useful if we assume Maxwell particles, first introduced for granular materials in ref. 1 and used in (2.1). The reason why this occurs is that when we try to form the equations satisfied by the moments, the contribution from the collision term contains a finite number of moments of order not higher than the order of the moment arising from the time derivative term in the

Boltzmann equation. This property was discovered by Maxwell and is characteristic of the particles that are now called after him.

As shown by Galkin[8] and TruesdellTr, the second order moment equations for a Maxwell gas, associated with a homoenergetic affine flow, are decoupled from those of higher order and can be solved explicitly. This result generalizes the result mentioned above for homoenergetic dilatations. In order to obtain the moment equations, one multiplies (2.23) by the appropriate monomial $c_{i_1} c_{i_2} \cdots c_{i_n}$, and integrate over the velocity space. The term with the derivative with respect to the velocity variables must be handled by partial integration; the collision term is complicated unless we adopt Maxwell particles, as we shall do. General results and details about the moment equations can be found in a recent paper.[2]

The most interesting moment equations are those for $p_{22} = \int c_1^2 f \, d\mathbf{v}$, $p_{12} = \int c_1 c_2 f \, d\mathbf{v}$, $p = \frac{1}{3} \int |\mathbf{c}|^2 \, f \, d\mathbf{v} = \rho\theta$:

$$\dot{p} + \tfrac{2}{3} K p_{12} + \varepsilon(1-\varepsilon) \, B(\rho) \sqrt{\theta} \; p = 0 \tag{3.1}$$

$$\dot{p}_{12} + \tfrac{1}{2}(1-\varepsilon^2) \, B(\rho) \sqrt{\theta} \; p_{12} + K p_{11} = 0 \tag{3.2}$$

$$\dot{p}_{11} + \tfrac{1}{2}(1-\varepsilon^2) \, B(\rho) \sqrt{\theta} \; (p_{11} - p) = 0 \tag{3.3}$$

Equations (3.1)–(3.3) form a system of three nonlinear first order differential equations that possesses a steady solution with $p_{11} = p$ and

$$p_{12} = -\frac{3}{2} \frac{1}{K} \varepsilon(1-\varepsilon) \, B(\rho) \sqrt{\theta} \; p \tag{3.4}$$

$$\theta = \frac{4}{3} \frac{1}{(1-\varepsilon^2) \, \varepsilon(1-\varepsilon)[B(\rho)]^2} K^2 \tag{3.5}$$

and hence

$$p_{12} = -\frac{3}{2} \frac{1}{K} \varepsilon(1-\varepsilon) B(\rho) \, \rho \theta^{3/2} \tag{3.6}$$

Hence

$$p_{12} = -\eta(\varepsilon) \, \rho [B(\rho)]^{-2} \, K \, |K| \tag{3.7}$$

where $\eta(\varepsilon)$ is a constant which, of course, tends to infinity when ε vanishes. Also the "temperature" of the granular material θ reaches a well defined value proportional to K^2.

4. STABILITY

It would be interesting to investigate the stability of the solution found above with respect to arbitrary perturbations. With the methods used in this paper we can only investigate it with respect to data which respect the assumption of homoenergetic affine flow. This is of course a big restriction, but may indicate a way to stabilize the flow. To simplify the problem we change the time unit by dividing the differential equations by $\frac{1}{2}(1-\varepsilon^2)\, B(\rho)\sqrt{\theta/p}$. We obtain:

$$\dot{p} + \tfrac{2}{3} Hq + \alpha\sqrt{p}\, p = 0 \tag{4.1}$$

$$\dot{q} + \sqrt{p}\, q + Hr = 0 \tag{4.2}$$

$$\dot{r} + \sqrt{p}\, r - \sqrt{p}\, p = 0 \tag{4.3}$$

where we have denoted $q = p_{12}$ and $r = p_{11}$

$$\alpha = \frac{2\varepsilon}{1+\varepsilon}$$

We study the stability by letting:

$$p = p_0 + x, \qquad q = q_0 + y, \qquad r = r_0 + z$$

where (p_0, q_0, r_0) denotes the steady solution. We have

$$\dot{x} + \frac{2}{3} Hy + \sqrt{\frac{3}{2}}\, \alpha |H|\, x = 0 \tag{4.4}$$

$$\dot{y} + \sqrt{\frac{2}{3\alpha}}\, |H|\, y - \frac{1}{2} Hx + Hz = 0 \tag{4.5}$$

$$\dot{z} + \sqrt{\frac{2}{3\alpha}}\, |H|\, z - \sqrt{\frac{2}{3\alpha}}\, |H|\, x = 0 \tag{4.6}$$

If we compute the eigenvalues associated with the exponential solutions behaving as $\exp(\lambda\, |H|\, t)$, we find that the eigenvalues satisfy

$$\lambda^3 + \left(\sqrt{\frac{3\alpha}{2}} + 2\sqrt{\frac{2}{3\alpha}}\right)\lambda^2 + \left(\frac{7}{3} + \frac{2}{3\alpha}\right)\lambda + \frac{2}{3}\sqrt{\frac{2}{3\alpha}} = 0$$

When e, and hence α, ranges between 0 and 1, this third degree equation has one negative real root and two complex conjugated roots with negative real part. Hence stability with respect to small perturbations in the restricted class investigated here follows.

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